Elastic torsion of a circular bar

This text is integrating part of the homonymous link in PEEI: a computer program for the numerical solution of systems of partial differential equations.

System of measurement: International System of Units, with the exception of the force that is expressed in $\mathrm{N}\times 10^{-12}$.

Coordinate system: Cartesian

Coordinates: $\underline{x}$ of which: $\underline{x}\equiv\{x_i;i=1,3\}$ $[x_i]\equiv$[length] $\underline{\Re}\langle x_i\rangle\equiv(-\infty,\infty)$, $\underline{x}$ a point of the deformed medium.

Coordinate versors: $\{\boldsymbol{\upsilon}_i;i=1,3\}$

Unknown functions: $\{s_1,s_2,s_3,\tau_{11},\tau_{12},\tau_{13},\tau_{22},\tau_{23},\tau_{33}\}$ of which: $s_i=x_i-X_i$, $[s_i]\equiv$[length], $\underline{X}\equiv\{X_i;i=1,3\}$, $\underline{X}$ the position of the point $\underline{x}$ in the undeformed medium, $\mathbf{s}\equiv\Sigma_{i=1,3}(s_i\cdot\boldsymbol{\upsilon}_i)$, $\mathbf{s}$ the displacement of the point $\underline{X}$, $\{\tau_{11},\tau_{12},\tau_{13},\tau_{22},\tau_{23},\tau_{33}\}$ the six independent components of the stress tensor, $[\tau_{ij}]\equiv$[stress], $\tau_{ij}=\tau_{ji}$.

Differential analytical model:

$\partial\tau_{11}(\underline{x})/\partial x_1+\partial\tau_{12}(\underline{x})/\partial x_2+\partial\tau_{13}(\underline{x})/\partial x_3+F_1(\underline{x})=0$

$\partial\tau_{12}(\underline{x})/\partial x_1+\partial\tau_{22}(\underline{x})/\partial x_2+\partial\tau_{23}(\underline{x})/\partial x_3+F_2(\underline{x})=0$

$\partial\tau_{13}(\underline{x})/\partial x_1+\partial\tau_{23}(\underline{x})/\partial x_2+\partial\tau_{33}(\underline{x})/\partial x_3+F_3(\underline{x})=0$

$\{(1+\nu)\cdot\tau_{ij}(\underline{x})-\delta_{ij}\cdot\nu\cdot(\tau_{11}(\underline{x})+\tau_{22}(\underline{x})+\tau_{33}(\underline{x}))-E\cdot(\partial s_i(\underline{x})/\partial x_j+\partial s_j(\underline{x})/\partial x_i)/2=0;j=i,3;i=1,3\}$

of which: $\mathbf{F}\equiv\Sigma_{i=1,3}(F_i\cdot\boldsymbol{\upsilon}_i)$, $\mathbf{F}$ the body force per unit volume, $\{\delta_{ij}=0;\forall i\neq j\}$ $\{\delta_{ij}=1;\forall i=j\}$, E Young's modulus, ν Poisson's ratio, $E=0.21$ $\nu=0.3$.

Related relations:

$\varepsilon_{ij}(\underline{x})=(\partial s_i(\underline{x})/\partial x_j+\partial s_j(\underline{x})/\partial x_i)/2=(1+\nu)\cdot\tau_{ij}(\underline{x})/E-\delta_{ij}\cdot\nu\cdot(\tau_{11}(\underline{x})+\tau_{22}(\underline{x})+\tau_{33}(\underline{x}))/E$ (1)

$\omega_{ij}(\underline{x})=(\partial s_i(\underline{x})/\partial x_j-\partial s_j(\underline{x})/\partial x_i)/2$ (2)

$T_i(\underline{x})=\Sigma_{j=1,3}(\tau_{ji}(\underline{x})\cdot\check{n}_j(\underline{x}))$ (3)

$s_i(\underline{x}_B)=s_i(\underline{x}_A)+\Sigma_{j=1,3}(\omega_{ij}(\underline{x}_A)\cdot(x_{Bj}-x_{Aj}))+\int_{A,B}(\Theta_i(c)\cdot dc)$ (4)

$\Theta_i(c)\equiv\Sigma_{j=1,3}(\varepsilon_{ij}(\underline{x}(c))\cdot x_j'(c)+(x_{Bj}-x_j(c))\cdot\Sigma_{k=1,3}((\partial\varepsilon_{ik}(\underline{x}(c))/\partial x_j-\partial\varepsilon_{jk}(\underline{x}(c))/\partial x_i)\cdot x_k'(c)))$ (5)

of which here, $\varepsilon_{ij}=\varepsilon_{ji}$, $\mathbf{T}(\underline{x})\equiv\Sigma_{i=1,3}(T_i(\underline{x})\cdot\boldsymbol{\upsilon}_i)$ $\check{\mathbf{n}}(\underline{x})\equiv\Sigma_{i=1,3}(\check{n}_i(\underline{x})\cdot\boldsymbol{\upsilon}_i)$, $\mathbf{T}(\underline{x})$ the stress vector in a point of a plane with normal outward versor $\check{\mathbf{n}}(\underline{x})$, $\underline{x}(c)\equiv\{x_i(c);i=1,3\}$ $\underline{x}_A\equiv\{x_{Ai};i=1,3\}=\underline{x}(A)$ $\underline{x}_B\equiv\{x_{Bi};i=1,3\}=\underline{x}(B)$

Definition set: $\{\underline{x} / x_1^2+x_3^2 \le R^2; 0 \le x_2 \le L_2\}$ $R=1/2$ $L_2=10$.

Conditions:

$F_1(\underline{x})=F_2(\underline{x})=F_3(\underline{x})=0$ $\{s_i(x_1,0,x_3)=0;i=1,3\}$ $\partial s_1(\underline{x}_A)/\partial x_2=\partial s_3(\underline{x}_A)/\partial x_2=0$ $\underline{x}_A\equiv\{0,0,0\}$ (6)

$T_1(x_1,0,x_3)=\mu\cdot\alpha\cdot x_3$ $T_2(x_1,0,x_3)=0$ $T_3(x_1,0,x_3)=-\mu\cdot\alpha\cdot x_1$ $T_1(x_1,L_2,x_3)=-\mu\cdot\alpha\cdot x_3$ $T_2(x_1,L_2,x_3)=0$ $T_3(x_1,L_2,x_3)=\mu\cdot\alpha\cdot x_1$ $\mu=E/(2\cdot(1+\nu))$ $\alpha=1$ (7)

In PEEI executions, the (7) is applied on the surface of the body that approximates the circular bar. This is coherent with the validity of this script for a bar with the cross section of every shape.

From (7) and (3) follows

$\tau_{12}(x_1,0,x_3)=\tau_{12}(x_1,L_2,x_3)=-\mu\cdot\alpha\cdot x_3$ $\tau_{22}(x_1,0,x_3)=\tau_{22}(x_1,L_2,x_3)=0$ $\tau_{23}(x_1,0,x_3)=\tau_{23}(x_1,L_2,x_3)=\mu\cdot\alpha\cdot x_1$

Related files: mad.txt

Exact solution:

From previous conditions follows $\tau_{11}(\underline{x})=\tau_{13}(\underline{x})=\tau_{22}(\underline{x})=\tau_{33}(\underline{x})=0$ $\tau_{12}(\underline{x})=-\mu\cdot\alpha\cdot x_3$ $\tau_{23}(\underline{x})=\mu\cdot\alpha\cdot x_1$. These and (1) imply

$\varepsilon_{11}(\underline{x})=\varepsilon_{22}(\underline{x})=\varepsilon_{33}(\underline{x})=\varepsilon_{13}(\underline{x})=0$ $\varepsilon_{12}(\underline{x})=-\alpha\cdot x_3/2$ $\varepsilon_{23}(\underline{x})=\alpha\cdot x_1/2$ (8)

From (2) and (6) follows $\omega_{ij}(\underline{x}_A)=0$. This, $\{s_i(\underline{x}_A)=0;i=1,3\}$ and (4) imply

$s_i(\underline{x}_B)=\int_{A,B}(\Theta_i(c)\cdot dc)$ (9)

Are placed

$\int_{A,B}(\Theta_i(c)\cdot dc)=\int_{A,P}(\Theta_i(c)\cdot dc)+\int_{P,Q}(\Theta_i(c)\cdot dc)+\int_{Q,B}(\Theta_i(c)\cdot dc)$ $\underline{x}(P)\equiv\{0,x_{B2},0\}$ $\underline{x}(Q)\equiv\{x_{B1},x_{B2},0\}$
$\{x_1'(c)=x_3'(c)=0,x_2'(c)=1;\forall c\in[A,P]\}$ $\{x_2'(c)=x_3'(c)=0,x_1'(c)=1;\forall c\in[P,Q]\}$
$\{x_1'(c)=x_2'(c)=0,x_3'(c)=1;\forall c\in[Q,B]\}$ (10)

These, (5) and (8) imply

$\{\Theta_1(c)=\alpha\cdot(x_3(c)/2-x_{B3}),\Theta_2(c)=0,\Theta_3(c)=\alpha\cdot(x_{B1}-x_1(c)/2);\forall c\in[A,P]\}$
$\{\Theta_1(c)=0,\Theta_2(c)=-\alpha\cdot x_{B3}/2,\Theta_3(c)=\alpha\cdot(x_{B2}-x_2(c))/2;\forall c\in[P,Q]\}$
$\{\Theta_1(c)=-\alpha\cdot(x_{B2}-x_2(c))/2,\Theta_2(c)=\alpha\cdot x_{B1}/2,\Theta_3(c)=0;\forall c\in[Q,B]\}$

From these, (9) and (10) follows

$s_1(\underline{x})=-\alpha\cdot x_2\cdot x_3$ $s_2(\underline{x})=0$ $s_3(\underline{x})=\alpha\cdot x_1\cdot x_2$ (11)

The (11) is valid for the elastic torsion of a bar with the cross section of every shape.

Note: In the following diagrams, the symbols + (plus), □ (empty square) and ■ (full square) are respectively inherent to $\underline{x}$, $\underline{X}$ determined by means of $X_i=x_i-s_i$ and (11), and $\underline{X}$ determined by means of $X_i=x_i-s_i$ where s_i is calculated by PEEI.

Case 5-3-5: points-5-3-5.txt, mem-5-3-5.bin, cond-5-3-5.txt, sol-5-3-5.txt, plot-5-3-5-1.jpg, plot-5-3-5-2.jpg, plot-5-3-5-3.jpg

Case 5-6-5: points-5-6-5.txt, mem-5-6-5.bin, cond-5-6-5.txt, sol-5-6-5.txt, plot-5-6-5-1.jpg, plot-5-6-5-2.jpg, plot-5-6-5-3.jpg

Case 5-9-5: points-5-9-5.txt, mem-5-9-5.bin, cond-5-9-5.txt, sol-5-9-5.txt, plot-5-9-5-1.jpg, plot-5-9-5-2.jpg, plot-5-9-5-3.jpg

Case 5-11-5: points-5-11-5.txt, mem-5-11-5.bin, cond-5-11-5.txt, sol-5-11-5.txt, plot-5-11-5-1.jpg, plot-5-11-5-2.jpg, plot-5-11-5-3.jpg

Case 7-3-7: points-7-3-7.txt, mem-7-3-7.bin, cond-7-3-7.txt, sol-7-3-7.txt, plot-7-3-7-1.jpg, plot-7-3-7-2.jpg, plot-7-3-7-3.jpg

Case 7-6-7: points-7-6-7.txt, mem-7-6-7.bin, cond-7-6-7.txt, sol-7-6-7.txt, plot-7-6-7-1.jpg, plot-7-6-7-2.jpg, plot-7-6-7-3.jpg

Case 7-9-7: points-7-9-7.txt, mem-7-9-7.bin, cond-7-9-7.txt, sol-7-9-7.txt, plot-7-9-7-1.jpg, plot-7-9-7-2.jpg, plot-7-9-7-3.jpg

Case 7-11-7: points-7-11-7.txt, mem-7-11-7.bin, cond-7-11-7.txt, sol-7-11-7.txt, plot-7-11-7-1.jpg, plot-7-11-7-2.jpg, plot-7-11-7-3.jpg

Case 9-3-9: points-9-3-9.txt, mem-9-3-9.bin, cond-9-3-9.txt, sol-9-3-9.txt, plot-9-3-9-1.jpg, plot-9-3-9-2.jpg, plot-9-3-9-3.jpg

Case 9-6-9: points-9-6-9.txt, mem-9-6-9.bin, cond-9-6-9.txt, sol-9-6-9.txt, plot-9-6-9-1.jpg, plot-9-6-9-2.jpg, plot-9-6-9-3.jpg

Case 9-9-9: points-9-9-9.txt, mem-9-9-9.bin, cond-9-9-9.txt, sol-9-9-9.txt, plot-9-9-9-1.jpg, plot-9-9-9-2.jpg, plot-9-9-9-3.jpg

Case 9-11-9: points-9-11-9.txt, mem-9-11-9.bin, cond-9-11-9.txt, sol-9-11-9.txt, plot-9-11-9-1.jpg, plot-9-11-9-2.jpg, plot-9-11-9-3.jpg

Bibliography:

[1] YU. A. AMENZADE, *Theory of Elasticity*, Mir Publishers, 1979, Moscow